SOFR So Far:
Modeling the LIBOR Replacement

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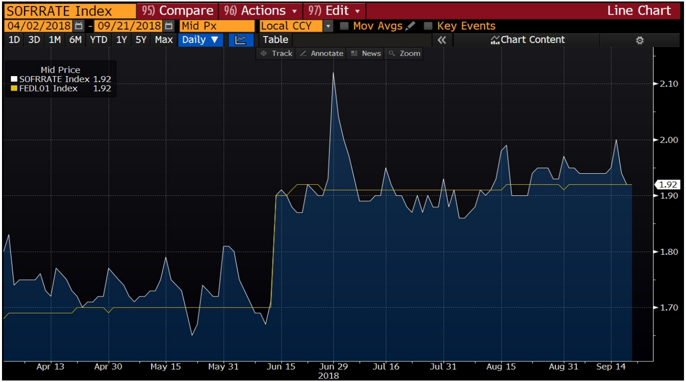
Introduction

- On June 22, 2017, the Alternative Reference Rates Committee (ARRC) identified a treasuries repo financing rate, which they called SOFR, as the best replacement for LIBOR.
- On July 27, 2017, Andrew Bailey, the head of the Financial Conduct Authority, said that LIBOR is not sustainable because of a lack of transactions providing data, and that it will be phased out in 2021.
- On November 29, 2017, Mark Carney, the Bank of England (BOE) Governor, said that the BOE has stepped up efforts to replace LIBOR with SONIA by the end of 2021.
- On July 12, 2018, Andrew Bailey announced that banks should look to move off of LIBOR sooner than 2021. He also said that interest rate derivatives do not need term rates, and that synthetic solutions created to replace LIBOR were not viable.

Introduction

- On each business day, starting April 2, 2018, the New York Fed has been publishing the SOFR on the New York Fed website.
- The SOFR is calculated as a volume-weighted median of transaction-level tri-party repo data collected from the Bank of New York Mellon as well as GCF Repo transaction data and data on bilateral Treasury repo transactions cleared through FICC's DVP service.



Introduction

- On May 7, 2018, CME launched 1-month and 3-month SOFR futures contracts.
- The contract listings of the 1-month futures comprise the nearest 7 calendar months.

Month	Charts	Last	Change	Prior Settle	Open	High	Low	Volume	Updated
SEP 2018		**98.01**	-0.005	98.015	98.0125	98.0125	98.01	126	14:10:59 CT 21 Sep 2018
OCT 2018		**97.79**	-0.005	97.795	97.79	97.79	97.79	199	14:10:59 CT 21 Sep 2018
NOV 2018		**97.775**	-0.005	97.78	97.775	97.775	97.775	31	14:10:59 CT 21 Sep 2018
DEC 2018		**97.695**	-0.005	97.70	97.695	97.695	97.695	84	14:10:59 CT 21 Sep 2018
JAN 2019		**97.58**	-0.01	97.59	97.58	97.58	97.58	17	14:10:59 CT 21 Sep 2018
FEB 2019		**-**	-	97.57	-	-	-	0	14:10:59 CT 21 Sep 2018
MAR 2019		**-**	-	97.515	-	-	-	0	14:10:59 CT 21 Sep 2018
APR 2019		**-**	-	0.00	-	-	-	0	16:45:00 CT 20 Sep 2018

Legend: OPT Options   Price Chart   ? About This Report

Introduction

- The contract listings of the 3-month futures comprise the 20 March quarterly months, which started with June 2018.

Month	Charts	Last	Change	Prior Settle	Open	High	Low	Volume	Updated
SEP 2018		-	-	97.795	-	-	-	0	14:11:06 CT 21 Sep 2018
DEC 2018		**97.57**	UNCH	97.57	97.575	97.575	97.57	70	15:31:35 CT 21 Sep 2018
MAR 2019		-	-	97.38	-	-	-	1	14:48:34 CT 21 Sep 2018
JUN 2019		**97.245**	**+0.015**	97.23	97.235	97.245	97.235	56	15:31:35 CT 21 Sep 2018
SEP 2019		**97.15**	**+0.01**	97.14	97.145	97.15	97.145	334	14:11:06 CT 21 Sep 2018
DEC 2019		-	-	97.085	-	-	-	254	15:33:48 CT 21 Sep 2018
MAR 2020		-	-	97.065	-	-	-	125	15:33:48 CT 21 Sep 2018
JUN 2020		-	-	97.065	-	-	-	50	14:11:06 CT 21 Sep 2018
SEP 2020		-	-	97.075	-	-	-	0	14:11:06 CT 21 Sep 2018
DEC 2020		-	-	97.07	-	-	-	0	14:11:06 CT 21 Sep 2018

Introduction

- On July 16, 2018, LCH cleared first SOFR-based derivatives.
- The first trade was a SOFR-Fed-funds basis swap arranged by TP ICAP.
- The trade was followed by a SOFR swap and another SOFR-Fed-funds basis swap.
- On July 26, 2018, Fannie Mae issued the market's first-ever SOFR FRNs.
- In September, CME will begin clearing SOFR fixed/float and basis swaps.
- Caveat: LCH decided to use Fed-fund for PAI, CME will instead use SOFR.
- Different collateral rates imply valuations based on different discount curves.

SOFR-linked product issuance

ISSUER	DATE	TYPE
Fannie Mae	07/26/2018	FRN
World Bank	08/14/2018	FRN
Credit Suisse	08/20/2018	CD
Barclays	08/24/2018	CP
MetLife	08/30/2018	FRN
MTA	09/20/2018	FRN

A minimal multi-curve model

- Our purpose is to build a SOFR curve using quoted SOFR derivative data.
- To this end, we introduce a simple multi-curve model, which extends that of Moreni and Pallavicini (2010), to simultaneously price SOFR futures and Eurodollar futures.
- We assume that:
  - OIS rates evolve according to the Hull-White one-factor (1990) model
  - The SOFR-OIS basis is deterministic
  - Forward LIBORs follow a shifted-lognormal LMM
- We then assume OIS discounting, and that numeraires are defined by the OIS curve.
- SOFR discounting can be modeled as well.

A minimal multi-curve model

- The instantaneous OIS short rate $r(t)$ is assumed to follow, under the risk-neutral measure Q, the Hull-White one-factor (1990) model

$$r(t) = x(t) + \alpha(t)$$

where α is a deterministic function, and

$$\mathrm{d}x(t) = -ax(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}Z(t), \quad x(0) = 0,$$

where a is a positive constant and σ is deterministic.

- We model the instantaneous SOFR short rate $s(t)$ by assuming that

$$s(t) = x(t) + \beta(t)$$

where β is deterministic and used to calibrate the time-0 SOFR curve.

- This implies that

$$s(t) - r(t) = \beta(t) - \alpha(t) =: \gamma(t)$$

A minimal multi-curve model

- Let us denote the price at time t of the OIS zero-coupon bond with maturity T by $P(t,T)$, that is:

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(u)\,\mathrm{d}u} | \mathcal{F}_t\right]$$

- The SOFR zero-coupon bond at time t with maturity T is then defined by:

$$\begin{aligned} P_s(t,T) &= \mathbb{E}\left[e^{-\int_t^T s(u)\,\mathrm{d}u} | \mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-\int_t^T r(u)\,\mathrm{d}u} e^{-\int_t^T \gamma(u)\,\mathrm{d}u} | \mathcal{F}_t\right] \\ &= P(t,T)\,\mathbb{E}^T\left[e^{-\int_t^T \gamma(u)\,\mathrm{d}u} | \mathcal{F}_t\right] \end{aligned}$$

- Since γ is assumed to be deterministic, then

$$P_s(t,T) = e^{-\int_t^T \gamma(u)\,\mathrm{d}u}\,P(t,T)$$

A minimal multi-curve model

- Assuming continuous compounding, we define the simply-compounded SOFR forward rate $F_j^s(t)$ for the interval $[T_{j-1}, T_j)$ by

$$1 + \tau_j F_j^s(t) = \mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} s(u)\,\mathrm{d}u} \middle| \mathcal{F}_t \right]$$

which can be expressed as

$$\begin{aligned}
1 + \tau_j F_j^s(t) &= \frac{1}{P(t,T_j)} \mathbb{E}\left[e^{-\int_t^{T_j} r(u)\,\mathrm{d}u} e^{\int_{T_{j-1}}^{T_j} s(u)\,\mathrm{d}u} \middle| \mathcal{F}_t \right] \\
&= \frac{1}{P(t,T_j)} \mathbb{E}\left[e^{-\int_t^{T_{j-1}} r(u)\,\mathrm{d}u} e^{\int_{T_{j-1}}^{T_j} \gamma(u)\,\mathrm{d}u} \middle| \mathcal{F}_t \right] \\
&= \frac{P(t,T_{j-1})}{P(t,T_j)} \mathbb{E}^{T_{j-1}}\left[e^{\int_{T_{j-1}}^{T_j} \gamma(u)\,\mathrm{d}u} \middle| \mathcal{F}_t \right]
\end{aligned}$$

- Since γ is deterministic, then

$$1 + \tau_j F_j^s(t) = \frac{P_s(t,T_{j-1})}{P_s(t,T_j)}$$

A minimal multi-curve model

- Forward LIBORs L_j, $j = 1, \ldots, n$, are defined by:

$$L_j(t) = \mathbb{E}^{T_j}\left[L(T_{j-1}, T_j)|\mathcal{F}_t\right]$$

  where $L(t, T)$ is the time-t spot LIBOR with maturity T.

- L_j is assumed to evolve under the T_j-forward measure according to:

$$\mathrm{d}L_j(t) = \sigma_j(t)[L_j(t) + \alpha_j]\,\mathrm{d}W_j(t)$$

  where σ_j is deterministic and α_j is constant, for all j.

- We assume a one-factor model for simplicity, that is we set $\mathrm{d}W_i(t)\,\mathrm{d}W_j(t) = \mathrm{d}t$ for all i, j.

- Forwards L_j are shifted-lognormal also under Q. So, Eurodollar futures can be priced in closed form, see Henrard (2014) and Mercurio (2018).

The model calibration

- The LIBOR shift parameters α_j can be set to be $\alpha_j = 1/\tau_j$. Alternatively, they can be calibrated to the corresponding caplet skews.
- The LIBOR volatilities σ_j can be calibrated to caplets ATM volatilities σ_j^{ATM}. Assuming constant σ_j, and lognormal σ_j^{ATM}, we have:

$$\sigma_j \approx \frac{\sigma_j^{\mathrm{ATM}} L_j(0)}{L_j(0) + \alpha_j}$$

- The LIBOR-OIS correlation can be calibrated to Eurodollar futures, or in a way to maximize smoothness of the corresponding LIBOR curve.
- The OIS-SOFR volatility can be defined so that a given LIBOR-OIS basis has minimal volatility, see Mercurio (2018).
- Alternatively, the OIS-SOFR volatility can be fine tuned to maximize smoothness of the corresponding SOFR curve.

The pricing of CME 1m-SOFR futures

- We consider a 1m-SOFR futures contract with maturity T, and whose delivery month is represented by $[T-\delta, T)$, where $\delta \approx 1/12$.
- We approximate the arithmetic average of daily SOFR during the delivery month by:

$$\frac{1}{\delta}\int_{T-\delta}^{T} s(u)\,\mathrm{d}u$$

- Assuming $T-\delta \geq 0$, the 1m SOFR futures rate $f^s(0; T-\delta, T)$ is calculated as follows:

$$\begin{aligned} f^s(0; T-\delta, T) &= \frac{1}{\delta}\mathbb{E}\left[\int_{T-\delta}^{T} s(u)\,\mathrm{d}u\right] \\ &= \frac{1}{\delta}\ln\frac{P_s(0, T-\delta)}{P_s(0,T)} + C^{1m}(0; T-\delta, T) \end{aligned}$$

  where $C^{1m}(0; T-\delta, T)$ is the 1m-SOFR futures convexity adjustment, which is an analytic function of a and $\sigma(t)$.

The pricing of CME 1m-SOFR futures

- Equivalently, we can write:

$$f^s(0;T-\delta,T) = R_s(0;T-\delta,T) + C^{1m}(0;T-\delta,T)$$

where $R_s(0;T-\delta,T)$ denotes the continuously-compounded SOFR forward rates for the interval $[T-\delta,T)$.

- In the case of a constant $\sigma(t)\equiv\sigma$, the convexity adjustment is explicitly given by

$$\begin{aligned} C^{1m}(0;T-\delta,T) &= \frac{\sigma^2}{2\delta a^2}\left[\delta + \frac{2}{a}e^{-aT}(1-e^{a\delta}) - \frac{1}{2a}e^{-2aT}(1-e^{2a\delta})\right] \\ &= \frac{\sigma^2}{6}[3T^2 - 3T\delta + \delta^2] + O(a) \end{aligned}$$

- Since $\delta\approx 1/12$, the maximum $T\approx 7/12$ and σ typically below 1%, then $C^{1m}(0;T-\delta,T)$ is likely to be a fraction of a basis point.

The pricing of CME 3m-SOFR futures

- We consider a 3m-SOFR futures contract with maturity T_j, and whose reference quarter is represented by the interval $[T_{j-1}, T_j)$.
- We approximate the compounded daily SOFR interest rate during the reference quarter by:

$$\frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{T_j} s(u)\,\mathrm{d}u} - 1\right]$$

- The 3m-SOFR futures rate $f_j^s(0)$ is calculated as follows:

$$1 + \tau_j f_j^s(0) = \mathbb{E}\left[e^{\int_{T_{j-1}}^{T_j} s(u)\,\mathrm{d}u}\right]$$

- Assuming $T_{j-1} \geq 0$, we have:

$$1 + \tau_j f_j^s(0) = \frac{P_s(0, T_{j-1})}{P_s(0, T_j)} e^{U_j}$$

  where U_j is an analytic function of a and $\sigma(t)$.

The pricing of CME 3m-SOFR futures

- The SOFR forward rate $F_j^s(0)$ can then be obtained from the quoted futures rate $f_j^s(0)$ as follows:

$$F_j^s(0) = \left[\frac{1}{\tau_j} + f_j^s(0)\right] e^{-U_j} - \frac{1}{\tau_j}$$

so, the 3m SOFR futures convexity adjustment is given by

$$C_j^{3m}(0) := f_j^s(0) - F_j^s(0) = \left[\frac{1}{\tau_j} + f_j^s(0)\right]\left(1 - e^{-U_j}\right)$$

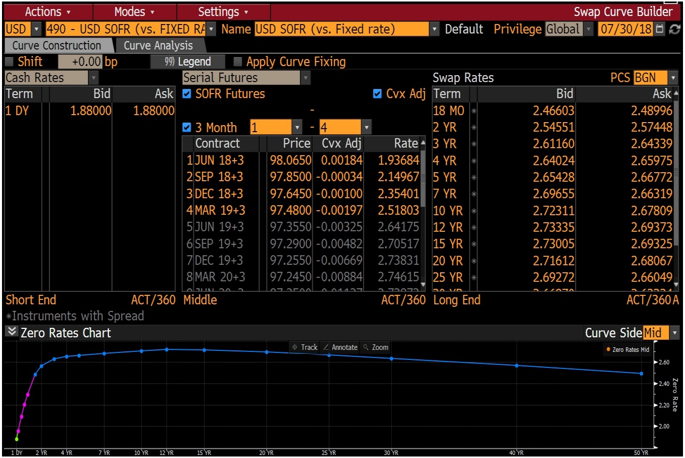
- In the case of a constant $\sigma(t) \equiv \sigma$, U_j is explicitly given by:

$$\begin{aligned} U_j &= \frac{\sigma^2}{2a^3}\left[e^{-a(T_j+T_{j-1})} - e^{-2aT_j} + e^{-a(T_j-T_{j-1})} + \dots\right] \\ &= \frac{\sigma^2}{6}\left[2T_j^3 - 3T_jT_{j-1}^2 + T_{j-1}^3\right] + O(a) \end{aligned}$$

- Since $\frac{1}{\tau_j} >> f_j^s(0)$ and U_j is small, then $C_j^{3m}(0) \approx \frac{1}{\tau_j} U_j \approx \frac{\sigma^2}{2} T_j^2$.

Stripping discount factors from futures

- SOFR discount factors $P_s(0,T)$ can be stripped from 1m and 3m futures rates using the previous formulas.
- A SOFR curve can then be extrapolated by assuming, for instance, a deterministic basis between SOFR and OIS swap rates.



The valuation of a SOFR fixed-floating swap

- Consider a swap where the floating leg pays at times T_j, $j = a+1, \dots, b$, and where the fixed leg pays the fixed rate K on dates $T'_{c+1}, \dots, T'_d$, with $T'_c = T_a$ and $T'_d = T_b$.
- The floating-leg payment at time T_j is approximately given by

$$e^{\int_{T_{j-1}}^{T_j} s(u)\,\mathrm{d}u} - 1$$

- The value of this payment at time $t \le T_{j-1}$ is

$$P(t,T_j)\,\mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} s(u)\,\mathrm{d}u} - 1 | \mathcal{F}_t\right] = \tau_j P(t,T_j) F_j^s(t)$$

- Then, the SOFR swap value to the fixed-rate payer, at time $t \le T_a$, is

$$\sum_{j=a+1}^{b} \tau_j P(t,T_j) F_j^s(t) - K \sum_{j=c+1}^{d} \tau'_j P(t,T'_j)$$

where τ'_j denotes the year fraction for the fixed-leg interval $[T'_{j-1}, T'_j)$.

The valuation of a SOFR fixed-floating swap

- The corresponding forward swap rate is then given by:

$$S(t) = \frac{\sum_{j=a+1}^{b} \tau_j P(t, T_j) F_j^s(t)}{\sum_{j=c+1}^{d} \tau_j' P(t, T_j')}$$

- When $T_a < t \leq T_{a+1}$, $\int_{T_a}^{t} s(u)\,\mathrm{d}u$ is known, so formulas must be modified accordingly.

- Equivalent formulas can be derived under SOFR discounting.

- CME is switching to SOFR discounting.

- LCH stays with Fed funds.

- Alternative SOFR fixed-floating swaps could be offered to please the buy side.

The new valuation of a LIBOR fixed-floating swap

- Consider a standard LIBOR-based swap where the floating leg pays at times T_j, $j = a+1, \ldots, b$, and where the fixed leg pays the fixed rate K on dates $T'_{c+1}, \ldots, T'_d$. We set $T'_c = T_a$ and $T'_d = T_b$.
- The swap value to the fixed-rate payer at time $t < T_{a+1}$ is given by

$$\sum_{j=a+1}^{b} \tau_j P(t, T_j) L_j(t) - K \sum_{j=c+1}^{d} \tau'_j P(t, T'_j)$$

  where we set $L_j(t) = L(T_a, T_{a+1})$ if $T_a \leq t < T_{a+1}$.
- This valuation relies on LIBOR being published at least until the last LIBOR fixing date T_{b-1}, so that forwards $L_j(t)$ can be defined accordingly.
- However, soon enough this may no longer be the case, because LIBOR is very likely to be discontinued before the end of 2021.

The new valuation of a LIBOR fixed-floating swap

- If LIBOR is to be discontinued, then swaps like the above are standard up to some payment time T_k (included), and from T_k (excluded) on they become swaps written on a new interest rate index.

- Assuming $T_k > T_a$, the valuation of the above swap must then be modified as follows:

$$\sum_{j=a+1}^{k} \tau_j P(t, T_j) L_j(t) + \sum_{j=k+1}^{b} \tau_j P(t, T_j) \hat{L}_j(t) - K \sum_{j=c+1}^{d} \tau_j' P(t, T_j')$$

  where $\hat{L}_j(t)$ denotes the forward at time t of the new LIBOR fallback $\hat{L}(T_{j-1}, T_j)$, that is:

$$\hat{L}_j(t) = \mathbb{E}^{T_j}\left[\hat{L}(T_{j-1}, T_j) | \mathcal{F}_t\right]$$

- The methodology for the new LIBOR fallback $\hat{L}(T_{j-1}, T_j)$ has not been decided yet, but ISDA started a consultation.

The new valuation of a LIBOR fixed-floating swap

- The LIBOR fallback will likely be defined as the sum of a SOFR-based term rate $R(T_{j-1}, T_j)$ and a LIBOR-SOFR basis spread $\mathcal{S}(T^*)$ calculated at the time $T^* < T_k$ when an official announcement of LIBOR discontinuation will be given.

- Therefore, we can write:

$$\hat{L}_j(t) = R_j(t) + \mathbb{E}^{T_j}\left[\mathcal{S}(T^*)|\mathcal{F}_t\right]$$

  where $R_j(t)$ is the time-t forward of $R(T_{j-1}, T_j)$.

- Forwards $\hat{L}_j(t)$ can be calculated using a multi-curve model where SOFR and LIBOR (and possibly OIS) rates are jointly modeled:
  - This will allow us to calculate $R_j(t)$, should the choice of term rate $R(T_{j-1}, T_j)$ generate a convexity adjustment for $R_j(t)$.
  - It will also allow us to calculate expected values of $\mathcal{S}(T^*)$, should it be modeled as stochastic.

The valuation of a LIBOR-SOFR basis swap

- A LIBOR-SOFR basis swap is a swap with two floating legs, one being the floating leg of a LIBOR fixed-floating swap, the other being the floating leg of a SOFR-based swap with the same maturity and payment frequency.
- Let us denote by T_a the start date of the swap, and by T_j, $j = a + 1, \ldots, b$ its payment dates.
- Assuming the same day count convention for the two legs, the value at time t of the basis swap to the LIBOR payer is:

$$\sum_{j=a+1}^{b} \tau_j P(t, T_j) F_j^s(t) - \sum_{j=a+1}^{k} \tau_j P(t, T_j) L_j(t) - \sum_{j=k+1}^{b} \tau_j P(t, T_j) \hat{L}_j(t)$$

where, for simplicity, we also assume $t \le T_a < T_k$.

Conclusions

- We have introduced a simple multi-curve model to price SOFR futures, as well as SOFR swaps, with the purpose of building a SOFR curve.
- We have also valued LIBOR based swaps under the new LIBOR fallback, and basis swaps.
- There are still many outstanding questions:
  - How will a risk-free term rate be calculated?
  - How will LIBOR fallbacks be defined?
  - Will there be LIBOR fallback bases?
  - Will SOFR-based derivatives be liquid enough?
  - Will there be a new LIBOR proxy?
  - Will there be a “zombie” LIBOR?
  - When will the market start to trade SOFR-based non-linear derivatives?
  - How to transition from a LIBOR-based contract to a SOFR-based one?
  - What about currencies other than USD, GBP, CHF, JPY and EUR?
  - ...

Appendix A: the minimal basis volatility

- We define the multiplicative LIBOR-OIS basis B_j as:

$$B_j(t) := \frac{L_j(t) - F_j(t)}{1 + \tau_j F_j(t)}$$

- The Q_j-dynamics of B_j is:

$$\begin{aligned} \mathrm{d}B_j(t) = \cdots\, \mathrm{d}t + \left[B_j(t) + \frac{1}{\tau_j}\right] \Bigg[& \sigma_j(t) \frac{L_j(t) + \alpha_j}{L_j(t) + \frac{1}{\tau_j}}\, \mathrm{d}W_j(t) \\ & - (B(t, T_j) - B(t, T_{j-1}))\sigma(t)\, \mathrm{d}Z_j(t) \Bigg] \end{aligned}$$

- Assuming a constant σ, minimizing the basis volatility of at time 0 yields:

$$\sigma = \frac{\rho \sigma_j(0)}{B(0, T_j) - B(0, T_{j-1})} \frac{L_j(0) + \alpha_j}{L_j(0) + \frac{1}{\tau_j}}$$

Appendix B: the pricing of Eurodollar futures

- The Eurodollar futures rate at time t for the same interval $[T_{j-1}, T_j)$ is defined by

$$f_j(t) = \mathbb{E}[L(T_{j-1}, T_j)|\mathcal{F}_t]$$

  and is associated with the Eurodollar-futures contract, with unit notional, that pays out $1 - L(T_{j-1}, T_j)$ at time T_{j-1}.

- The futures convexity adjustment is defined by:

$$C_j(t) = f_j(t) - L_j(t)$$

- In our simple multi-curve model, Eurodollar-futures convexity adjustments can be calculated exactly and in closed form:

$$C_j(0) = [L_j(0) + \alpha_j] \left[\exp\left(\rho \int_0^{T_{j-1}} \sigma_j(t)\sigma(t)B(t, T_j)\,\mathrm{d}t \right) - 1 \right]$$